3.2 Exercises

1. Find the derivative of $y = (x^2 + 1)(x^3 + 1)$ in two ways: by using the Product Rule and by performing the multiplication first. Do your answers agree?

2. Find the derivative of the function

$$F(x) = \frac{x - 3x\sqrt{x}}{\sqrt{x}}$$

in two ways: by using the Quotient Rule and by simplifying first. Show that your answers are equivalent. Which method do you prefer?

3–22 □ Differentiate.

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3. $f(x) = x^2e^x$

4. $g(x) = \sqrt{x}\, e^x$

5. $y = \dfrac{e^x}{x^2}$

6. $y = \dfrac{e^x}{1 + x}$

7. $h(x) = \dfrac{x + 2}{x - 1}$

8. $f(u) = \dfrac{1 - u^2}{1 + u^2}$

9. $G(s) = (s^2 + s + 1)(s^2 + 2)$

10. $g(x) = (1 + \sqrt{x})(x - x^3)$

11. $H(x) = (x^3 - x + 1)(x^{-2} + 2x^{-3})$

12. $H(t) = e^t(1 + 3t^2 + 5t^4)$

13. $y = \dfrac{3t - 7}{t^2 + 5t - 4}$

14. $y = \dfrac{4t + 5}{2 - 3t}$

15. $y = \dfrac{x^2 + 4x + 3}{\sqrt{x}}$

16. $y = \dfrac{\sqrt{x} - 1}{\sqrt{x} + 1}$

17. $y = (r^2 - 2r)e^r$

18. $y = \dfrac{u^2 - u - 2}{u + 1}$

19. $y = \dfrac{1}{x^4 + x^2 + 1}$

20. $y = \dfrac{e^x}{x + e^x}$

21. $f(x) = \dfrac{x}{x + \dfrac{c}{x}}$

22. $f(x) = \dfrac{ax + b}{cx + d}$

23–26 □ Find an equation of the tangent line to the curve at the given point.

23. $y = \dfrac{2x}{x + 1}$, $(1, 1)$

24. $y = \dfrac{\sqrt{x}}{x + 1}$, $(4, 0.4)$

25. $y = 2xe^x$, $(0, 0)$

26. $y = \dfrac{e^x}{x}$, $(1, e)$

27. (a) The curve $y = 1/(1 + x^2)$ is called a **witch of Maria Agnesi**. Find an equation of the tangent line to this curve at the point $\left(-1, \frac{1}{2}\right)$.

(b) Illustrate part (a) by graphing the curve and the tangent line on the same screen.

28. (a) The curve $y = x/(1 + x^2)$ is called a **serpentine**. Find an equation of the tangent line to this curve at the point $(3, 0.3)$.

(b) Illustrate part (a) by graphing the curve and the tangent line on the same screen.

29. (a) If $f(x) = e^x/x^3$, find $f'(x)$.

(b) Check to see that your answer to part (a) is reasonable by comparing the graphs of f and f'.

30. (a) If $f(x) = x/(x^2 - 1)$, find $f'(x)$.

(b) Check to see that your answer to part (a) is reasonable by comparing the graphs of f and f'.

31. Suppose that $f(5) = 1$, $f'(5) = 6$, $g(5) = -3$, and $g'(5) = 2$. Find the values of (a) $(fg)'(5)$, (b) $(f/g)'(5)$, and (c) $(g/f)'(5)$.

32. If $f(3) = 4$, $g(3) = 2$, $f'(3) = -6$, and $g'(3) = 5$, find the following numbers:

(a) $(f + g)'(3)$

(b) $(fg)'(3)$

(c) $(f/g)'(3)$

(d) $\left(\dfrac{f}{f - g}\right)'(3)$

33. If $f(x) = e^xg(x)$, where $g(0) = 2$ and $g'(0) = 5$, find $f'(0)$.

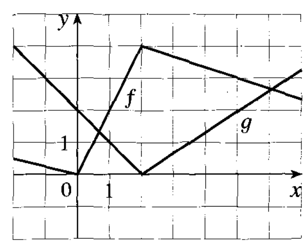
34. If $h(2) = 4$ and $h'(2) = -3$, find

$$\left.\frac{d}{dx}\left(\frac{h(x)}{x}\right)\right|_{x=2}$$

35. If f and g are the functions whose graphs are shown, let $u(x) = f(x)g(x)$ and $v(x) = f(x)/g(x)$.

(a) Find $u'(1)$.

(b) Find $v'(5)$.



36. If f is a differentiable function, find an expression for the derivative of each of the following functions.

(a) $y = x^2f(x)$

(b) $y = \dfrac{f(x)}{x^2}$

(c) $y = \dfrac{x^2}{f(x)}$

(d) $y = \dfrac{1 + xf(x)}{\sqrt{x}}$

37. In this exercise we estimate the rate at which the total personal income is rising in the Miami–Ft. Lauderdale metropolitan area. In July, 1993, the population of this area was 3,354,000, and the population was increasing at roughly 45,000 people per year. The average annual income was \$21,107 per capita, and this average was increasing at about \$1900 per year (well above the national average of about \$660 yearly). Use the

Product Rule and these figures to estimate the rate at which total personal income was rising in Miami–Ft. Lauderdale in July, 1993. Explain the meaning of each term in the Product Rule.

38. A manufacturer produces bolts of a fabric with a fixed width. The quantity q of this fabric (measured in yards) that is sold is a function of the selling price p (in dollars per yard), so we can write $q = f(p)$. Then the total revenue earned with selling price p is $R(p) = pf(p)$.

(a) What does it mean to say that $f(20) = 10{,}000$ and $f'(20) = -350$?

(b) Assuming the values in part (a), find $R'(20)$ and interpret your answer.

39. How many tangent lines to the curve $y = x/(x + 1)$ pass through the point $(1, 2)$? At which points do these tangent lines touch the curve?

40. Find equations of the tangent lines to the curve $y = (x - 1)/(x + 1)$ that are parallel to the line $x - 2y = 2$.

41. (a) Use the Product Rule twice to prove that if f, g, and h are differentiable, then

$$(fgh)' = f'gh + fg'h + fgh'$$

(b) Taking $f = g = h$ in part (a), show that

$$\frac{d}{dx}[f(x)]^3 = 3[f(x)]^2 f'(x)$$

(c) Use part (b) to differentiate $y = e^{3x}$.

42. (a) Use the definition of a derivative to prove the **Reciprocal Rule**: If g is differentiable, then

$$\frac{d}{dx}\left(\frac{1}{g(x)}\right) = -\frac{g'(x)}{[g(x)]^2}$$

(b) Use the Reciprocal Rule to differentiate the function in Exercise 19.

43. Use the Reciprocal Rule to verify that the Power Rule is valid for negative integers, that is,

$$\frac{d}{dx}(x^{-n}) = -nx^{-n-1}$$

for all positive integers n.

44. Use the Product Rule and the Reciprocal Rule to prove the Quotient Rule.

3.3 Rates of Change in the Natural and Social Sciences

Recall from Section 2.8 that if $y = f(x)$, then the derivative dy/dx can be interpreted as the rate of change of y with respect to x. In this section we examine some of the applications of this idea to physics, chemistry, biology, economics, and other sciences.

Let's recall from Section 2.7 the basic idea behind rates of change. If x changes from x_1 to x_2, then the change in x is

$$\Delta x = x_2 - x_1$$

and the corresponding change in y is

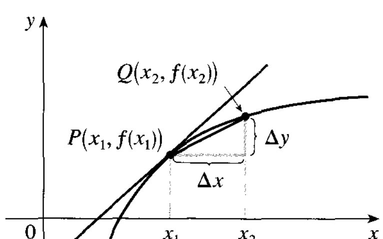
$$\Delta y = f(x_2) - f(x_1)$$

The difference quotient

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

is the **average rate of change of y with respect to x** over the interval $[x_1, x_2]$ and can be interpreted as the slope of the secant line PQ in Figure 1. Its limit as $\Delta x \to 0$ is the derivative $f'(x_1)$, which can therefore be interpreted as the instantaneous rate of change of y with respect to x or the slope of the tangent line at $P(x_1, f(x_1))$. Using Leibniz notation, we write the process in the form

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$



m_{PQ} = average rate of change
$m = f'(x_1)$ = instantaneous rate of change

FIGURE 1